Burgers Equation for Wave Propagation in an Infinite Stout-Wall Tube

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The reductive Taniuti-Wei method is used to obtain the Burgers equation for a wave propagating in an infinite stout-wall tube.

1. Introduction

Recently the Burgers equation for a wave propagating in a thin-wall tube has been found [1]. It seems to be more realistic to consider the irrotational motion of fluid in straight infinite stout-wall tube to take into account dissipation of energy and the nonlinearity of the compressible medium. The fluid wave equations are presented by

(i) the equation of continuity

$$[(h + B) r^2]_t + [(h + B) r^2 V]_x = 0, \qquad (1.1)$$

(ii) the Euler equation

$$d(h + B)(V_t + V V_x) + h_x - \mu C V_{xx} = 0, \quad (1.2)$$

(iii) the Newton equation (see Appendix)

$$\varrho_m (2r + h) h h_{tt} = 3 \left\{ (r + h) q - \frac{(h + B) r}{C} \right. \qquad (1.3)$$

$$\left. + \frac{h}{a^2 - b^2} \left[q b^2 - \frac{a^2 (h + B)}{C} + \frac{(q C - B - h) a^2 b^2}{C(r + h) r} \right] \right\}$$

where the subscripts x and t indicate partial differentiations, h is the wall thickness of the tube, V, μ, and ϱ are the velocity, viscosity and density of the liquid, respectively, p is the inside pressure of the liquid, r the inside radius of the tube, d the constant described by $\varrho \equiv d \cdot p$, ϱ_m the density of the material of the tube, q the outside pressure of the liquid, a, b the inside and outside radii of the tube in the undisturbed state, respectively,

$$H \equiv b - a,$$

$$C \equiv \frac{(1 - v) a^2 + (1 + v) a b}{E(a + b)},$$

$$B \equiv \frac{(1 - v) b^2 + (1 + v) a b}{E(a + b)} q - H,$$

E Young's modulus, and v the Poisson constant.
Using the following symbols

$$l_1 \equiv \varrho_m (a^2 - b^2) C, \quad l_2 \equiv q(a^2 - b^2) C,$$

$$l_3 \equiv a^2 - b^2, \qquad l_4 \equiv q b^2 C,$$

$$l_5 \equiv a^2 b^2 q C, \qquad l_6 \equiv a^2 b^2,$$

it is useful to rewrite (1.3) in the form

$$l_1 r h (r + h)(2r + h) h_{tt}$$

$$= 3 \{ r(r + h)[l_2(r + h) - l_3 r(h + B)] \qquad (1.4)$$

$$+ h[r(r + h)(l_4 - a^2(h + B)) + l_5 - l_6(B + h)] \}.$$

2. The Reductive Method

The co-called Taniuti-Wei method [2] is an attempt to reduce the above set of Eqs. (1.1), (1.2), (1.4) to a single equation describing the evolution of the parameters h, V, r. In order to arrive at this gool we introduce a small parameter ε and stretch the coodinates as follows:

$$\xi = \varepsilon (x + V_0 t), \quad \tau = \varepsilon^2 t. \qquad (2.1)$$

Having in mind the forms of the derivatives of ξ and τ, the fundamental set of equations can be rewritten as follows:

$$[(h + B) r^2 (V + V_0)]_\xi + \varepsilon [(h + B) r^2]_\tau = 0, \qquad (2.2)$$

$$d \cdot (h + B)[(V + V_0) V_\xi + \varepsilon V_\tau] + (h - \varepsilon \mu C V_\xi)_\xi = 0, \qquad (2.3)$$

$l_1 r h (r + h)(2r + h)$

$\qquad \cdot (\varepsilon^2 V_0^2 h_{\xi\xi} + 2\varepsilon^3 V_0 h_{\xi\tau} + \varepsilon^4 h_{\tau\tau})$

$= 3\{r(r + h)[l_2(r + h) - l_3 r(h + B)] \qquad (2.4)$

$\qquad + h[r(r + h)(l_4 - a^2(h + B)) + l_5 - l_6(B + h)]\}.$

Now we expand the dependent variables around the undisturbed uniform state as a power series in terms of ε:

$$r = a + \varepsilon r_1 + \varepsilon^2 r_2 + \dots, \qquad (2.5)$$

$$h = H + \varepsilon h_1 + \varepsilon^2 h_2 + \dots, \qquad (2.6)$$

$$V = V_0 + \varepsilon V_1 + \varepsilon^2 V_2 + \dots. \qquad (2.7)$$

Substituting these expressions into $(2.2) - (2.4)$ and equating coefficients of the same power in ε, we obtain

$$[2 V_0 h_1 + (H + B) V_1]_\xi = 0, \qquad (2.8)$$

$$[h_1 + 2d(H + B) V_0 V_1]_\xi = 0, \qquad (2.9)$$

$$L_1 r_1 + L_2 h_1 = 0, \qquad (2.10)$$

where

$L_1 \equiv a(a + H)[l_2 - l_3(H + B)] + (2a + H)$

$\qquad \cdot \{[l_2(a + H) - l_3 a(H + B)] + H[l_4 - a^2(H + B)]\},$

$L_2 \equiv a[(a + H)(2 l_2 - l_3 a) - l_3 a(H + B)]$

$\qquad + a[l_4 - a^2(H + B)](2H + a)$

$\qquad - H[a^3(a + H) + l_6] + l_5 - l_6(H + B).$

Taking into account these equations, we get

$$V_0^2 = \frac{1}{4d}, \qquad (2.11)$$

$$V_1 = R h_1, \qquad (2.12)$$

$$r_1 = P h_1. \qquad (2.13)$$

In (2.12) and (2.13) we have used the notations

$$R \equiv \frac{-2 V_0}{H + B}, \qquad P \equiv -\frac{L_2}{L_1}.$$

Equating the coefficients of the same powers in ε^2 in $(2.2) - (2.4)$, we find the following expressions:

$\frac{2 V_0(H + B)}{a^2} r_1^2 + 2 V_0 h_2$

$\qquad + (H + B)[V_2 + h_1 V_1]_\xi + h_{1\tau} = 0, \qquad (2.14)$

$(h_2 - \mu C V_{1\xi})_\xi$

$\qquad + d(H + B)[2 V_0 V_{2\xi} + V_1 V_{1\xi} + V_{1\tau}]$

$\qquad + 2d V_0 h_1 V_{1\xi} = 0, \qquad (2.15)$

$$S_1 r_1^2 + S_2 r_1 h_1 + S_3 r_2 + S_4 h_2 + S_5 h_1^2 = 0, \qquad (2.16)$$

where we used the symbols

$S_1 \equiv a l_2 - 2a l_3(H + B)$

$\qquad + [2 l_2 - l_3(H + B)](a + H) + [l_4 - a^2(H + B)] H,$

$S_2 \equiv (a + H)\{2 [l_4 - a^2(H + B)] - a^2 H - 2a l_3 + 2 l_2\}$

$\qquad - 2a l_3(H + B) + 2a l_2 - a^2 l_3 - a^3 H,$

$S_3 \equiv (a + H)\{2a [l_2 - l_3(H + B)] + (a + H) l_2\}$

$\qquad - l_3 a^2(H + B) + H[l_4 - a^2(H + B)](2a + H),$

$S_4 \equiv a(a + H)[l_2 - a l_3]$

$\qquad + a[l_2(a + H) - l_3 a(H + B)]$

$\qquad - l_6 H + l_5 - l_6(H + B) - a^3(a + H) H$

$\qquad + [l_4 - a^2(H + B)][aH + a(a + H)],$

$S_5 \equiv a l_2 - a^2 l_3 - l_6 - a^3 H - a^3(a + H)$

$\qquad + [l_4 - a^2(H + B)] a.$

Using (2.12) and (2.13), from $(2.14) - (2.16)$ we obtain

$$Q h_1^2 + S_3 r_2 + S_4 h_2 = 0, \qquad (2.17)$$

$$[G_1 P^2 h_1^2 + 2 V_0 h_2 + G_2 V_2 + R h_1^2]_\xi + h_{1\tau} = 0, \qquad (2.18)$$

$(h_2 - \mu C R h_{1\xi})_\xi + G_3 [2 V_0 V_{2\xi} + R^2 h_1 h_{1\xi} + R h_{1\tau}]$

$\qquad\qquad + 2d V_0 R h_1 h_{1\xi} = 0, \qquad (2.19)$

where

$$G_1 \equiv \frac{2 V_0(H + B)}{a^2}, \qquad G_2 \equiv H + B,$$

$$G_3 \equiv d \cdot G_2, \qquad Q = S_1 P^2 + S_2 P + S_5.$$

From these equations, having in mind (2.11), one can find a single equation in h_1, known as Burgers equation:

$$h_{1\tau} + \alpha\, h_1 h_{1\xi} + \beta h_{1\xi\xi} = 0. \qquad (2.20)$$

The nonlinear (α) and dissipative (β) coefficients are described by the formulas

$$\alpha \equiv \frac{G_2 W_2 - 2 V_0 G_3 W_1}{W_3}, \qquad \beta \equiv \frac{-\mu S_4 CR G_2}{W_3},$$

where

$$W_1 \equiv 2\,(S_1 P^2 + R)\,S_4 - 4\,Q V_0,$$

$$W_2 \equiv (2\,d \cdot V_0 + G_3 R)\,R S_4 - 2\,Q,$$

$$W_3 \equiv (G_2 R - 2\,V_0)\,G_3 S_4.$$

In a different context, the Burgers equation and its solutions have been discussed in some detail, see e.g. [3] and references.

The classical solution of this equation was found by the author using the bilinear Hirota method [4]:

$$h_1 = \frac{g_0 + Z_1 \exp \eta}{f_0 + Z_2 \exp \eta}, \qquad (2.21)$$

where

$$\eta = K\xi + \omega\tau,$$

$$K = \frac{\alpha\,(f_0 Z_1 - g_0 Z_2)}{2\beta f_0 Z_2}, \qquad \omega = \frac{\beta K^2\,(g_0 Z_2 + f_0 Z_1)}{g_0 Z_2 - f_0 Z_1},$$

and all the physical constants g_0, f_0, Z_1, Z_2 may be obtained from the initial conditions.

3. Summary

The discussion of an incompressible fluid that is confined within an infinitely long circular cylinder (with an elastic circular wall) leads to the Korteweg-de Vries equation [5]. It seems to be interesting to ask how the Korteweg-de Vries equation is modified if also effects of dissipation are taken into consideration [6]. The Korteweg-de Vries-Burgers equation has previously been derived in attempts to model the flow of blood through an artery.

In this paper a theory of nonlinear waves has been developed on the basis of the reductive method to obtain the Burgers equation for a wave propagation in an infinite stout-wall tube taking into account some physical effects such as nonlinearity and dissipation of the medium.

It is worth noticing that the formula (2.11), which describes the velocity of a linear wave, has a simpler form than the same formula for a wave in an infinite thin-wall tube.

Appendix

From the picture (Fig. 1) it follows that Newton equation takes the form

$$\varrho_m\,l\,\frac{R^2 - r^2}{2}\,\mathrm{d}\varphi \cdot \frac{2}{3}\,(R - r)_{tt}$$
$$= q\,l\,R\,\mathrm{d}\varphi - p\,l\,r\,\mathrm{d}\varphi + 2\,\sigma_e \sin \frac{\mathrm{d}\varphi}{2}\,(R - r)\,l, \quad (A.1)$$
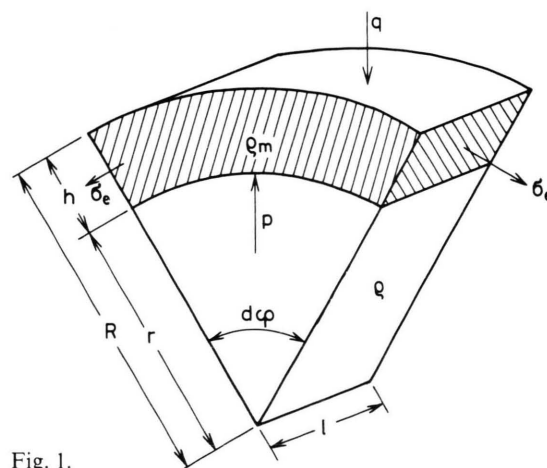

Fig. 1.

where σ_e is an average tension. After some calculations we obtain

$$\frac{\varrho_m}{3}\,(2\,r + h)\,h\,h_{tt} = q\,(r + h) - p\,r + \sigma_e\,h. \quad (A.2)$$

σ_e and h can be calculated via the Lamé method

$$\sigma_e = \frac{1}{a^2 - b^2}\left[q\,b^2 - a^2\,\frac{h + B}{C} + \frac{(q\,C - h - B)\,a^2\,b^2}{C\,(r + h)\,r}\right], \quad (A.3)$$

$$h = C\,p - B. \quad (A.4)$$

From (A.2), (A.3) and (A.4) one gets (1.3).

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